Derivation of Average Slope Formula

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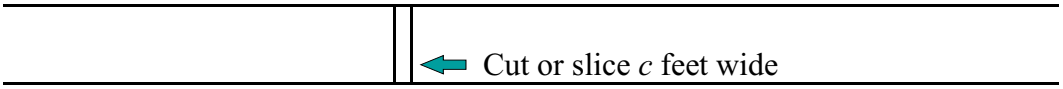
I will derive the average slope formula for the area between two contour lines.

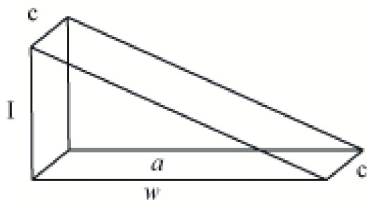
Contour line 1



Contour line 2

Standard Definitions:
I = space between the contour lines.
A = area between the two contour lines.
S = average slope between the two contour lines.
L = length of upper contour line (assume both are same length)

Side view of this slice.



Slope of slice is I/*w*
a = ground area of slice = *w* x *c*
So *w* = *a*/*c*
substituting, Slope = I/(*a*/*c*) = *c*I/*a*

Call *n* the number of slices of width c between these contour lines, and *L* the contour line length.
The length is the number of slices time the width of the slices: *L* = *nc*.
Dividing each side by *c*, we get n = *L*/*c*

Call S the average slope between the two contour lines.
This average slope is the sum of all the slopes of the slices, divided by the number of slices.
The math formula for this is:

$$S = \frac{\sum_{1}^{n} \frac{cI}{a}}{n}$$

This means add up all the slopes of the slices (*c*I/*a*), and then divide by the number of slices (*n*).

Let's start by solving the top part of this equation.

$$\sum \frac{cI}{a} = \frac{\sum c \sum I}{\sum a}$$

The sum of the c's, the length of the slices, is just L (length of the contour line).
If you sum up I (the contour interval) n times, you get nI.
The sum of the a's, the ground area of each slice, is A (total area between the two contour lines).
So,

$$\sum \frac{cI}{a} = \frac{\sum c \sum I}{\sum a} = \frac{LnI}{A}$$

Putting this in our original formula for slope, we get

$$S = \frac{LnI/A}{n} = \frac{LI}{A}$$

This is the average slope below the top contour line. If you add more contour lines, you just add all the average slopes together and divide by the number of times you do this, so you still have the same formula, S = LI/A.

Since this formula calculates the slope below each contour line, you shouldn't do this calculation for the last contour line. Therefore, many cities subtract one contour line by subtracting the length of an average contour line from L. If there is a contour line at the exact top and exact bottom of the lot, this works pretty well, but if there isn't, this approximation gives a slight benefit in slope calculation to the property owner.

Note: The formula above could be more rigorously derived using calculus, but I thought that would be more confusing to people reading this.